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AN ASYMPTOTIC APPROACH TO THE PROBLEMS OF THE THEORY OF ELASTICITY OF BODIES OF FINITE DIMENSIONS*

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A method is developed to solve the equilibrium problems of elastic bodies of fixed dimensions, based on a separation of the boundary-layer part of the solution by considering the problem for a half-strip. A closed solution in quadratures is constructed for the half-strip with a free lateral face and with given normal displaced longitudinal boundaries, using both symmetric and antisymmetric loading. When the normal stresses on the front boundaries are specified, the problem reduces to an integral equation of the first kind in a semi-infinite interval, the inversion of which is obtained by reduction to an infinite system of algebraic equations. The approach considered for problems of bodies of finite dimensions is asymptotic with respect to the small parameter characterizing the body's thickness. Testing of the method on a plane problem for an elastic rectangle enables the range of variation of this parameter to be investigated, in which this procedure is fairly accurate. In the example considered, for the case of a rectangular area, the stresses are found and compared with the results obtained earlier by other methods. The nature of the influence of the boundary layer on the stress distribution inside the body is investiaged.

Asymptotic methods, used for bodies of slab configuration, one of whose characteristic dimensions (thickness) is significantly less than the other two /1-4/, can obviously be classified into three types. The first of them /5,6/ is characterized by the application of joined asymptotic expansions to a certain class of solutions of the equations of the theory of elasticity, namely uniform solutions. The second type /7,8/ is distinguished by an asymptotic analysis of the equations of the theory of elasticity. From this it is clear that, to separate the boundary-layer part of the solution, it is enough, to a first approximation, to examine the two-dimensional problem and the problem of torsion for the half-strip, the lateral face of which is combined with the generating lateral surface of the plate at a given point. Finally, the third class includes the Vekua-Poniatovskii theory /9,10/, in which asymptotic methods are also developed /11/. In this sense this paper relates to the second of these methods.

1. In a Cartesian system of coordinates x, y we will examine the statical problem of the two-dimensional deformation of an elastic isotropic half-strip (the x-axis is the axis of symmetry and is directed parallel to the side faces, and the y-axis lies in the place of the half-strip end face) with the following boundary conditions:

$$x = 0,\ \tau_{xy} = 0,\ \sigma_x = 0 \qquad (1.1)$$
$$y = \pm 1,\ \tau_{xy} = 0,\ u_y = \pm f(x)$$

We will assume the boundary function $f(x)$ to be fairly smooth.

In this symmetric case, (the extension-compression case), the problem was examined in /12/, where its closed solution was obtained, based on the theory of dislocations. Here a similar result will be obtained using the well-known classical representation of the solution for a half-strip (ν is Poisson's ratio) /2/

$$u_x = -\frac{2}{\pi}\int_0^\infty \left[A(s)\operatorname{ch} sy + C(s)\left(\frac{3m-4}{ms}\operatorname{ch} sy + y\operatorname{sh} sy\right)\right]\sin sx\, ds + \sum_{n=1}^\infty (B_n + D_n x)\, e^{-\alpha_n x}\cos\alpha_n y$$

$$u_y = \frac{2}{\pi}\int_0^\infty [A(s)\operatorname{sh} sy + C(s)\, y\operatorname{ch} sy]\cos sx\, ds +$$

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$$\sum_{n=1}^{\infty}\left(B_n + D_n x - D_n \frac{3m-4}{m\alpha_n}\right) e^{-\alpha_n x}\sin\alpha_n y; \quad \alpha_n = \pi n, \quad m = \frac{1}{\nu}$$

By satisfying the boundary conditions (1.1), we obtain the relations

$$A(s)s = -C(s)\left(2\frac{m-1}{m} + s\,\mathrm{cth}\,s\right), \quad B_n\alpha_n = 2\frac{m-1}{m}D_n \tag{1.2}$$

$$C(s)\frac{\mathrm{sh}\,s}{s} = -\frac{m}{2(m-1)}\int_0^{\infty} f(t)\cos st\,dt$$

$$\sum_{n=1}^{\infty} D_n \cos\pi ny = -\frac{2}{\pi}\int_0^{\infty} C(s)\left[\mathrm{ch}\,sy\,(1 - s\,\mathrm{cth}\,s) + sy\,\mathrm{sh}\,sy\right] ds$$

Further, we find the Fourier series coefficients in the last relations of (1.2) and we use the formula for the function $C(s)$ from the third relation (1.2). Confining ourselves to functions $f(x)$, which tend to zero as $x\to\infty$, after integration along s [13] we obtain

$$D_n = (-1)^n \alpha_n \frac{m}{1-m}\int_0^{\infty} f'(t)\,t e^{-\alpha_n t}\,dt \tag{1.3}$$

Now for the Airy function $\psi(x, y)$ we have the following representation:

$$\frac{1}{2G}\frac{\partial\psi}{\partial y} = -\frac{1}{2G}\int\tau_{xy}\,dx = \frac{1}{\pi}\frac{m}{m-1}\int_0^{\infty} f'(t)\,dt\int_0^{\infty}\left(\frac{\mathrm{ch}\,s\,\mathrm{sh}\,sy}{\mathrm{sh}^2 s} - y\frac{\mathrm{ch}\,sy}{\mathrm{sh}\,s}\right)\sin st\cos sx\,ds + \tag{1.4}$$

$$\frac{\partial}{\partial y}\sum_{n=1}^{\infty} D_n\left(x + \frac{1}{\alpha_n}\right)e^{-\alpha_n x}\frac{\cos\alpha_n y}{\alpha_n}$$

The integral along s, in (1.4), is evaluated by integrating the first term by parts. To compute the series in (1.4) we use (1.3) and formulas for summing the resulting series /13/. As a result, integrating (1.4) along y, we finally obtain for the Airy function

$$\psi(x, y) = -\frac{2G}{1-\nu}\int_0^{\infty} f'(t)\,\Psi(x, y, t)\,dt \tag{1.5}$$

$$\Psi(x, y, t) = \frac{1}{2\pi}\left[\frac{t-x}{2}\ln\frac{\chi^-}{\chi^+} + \pi xt\frac{\mathrm{sh}\,\pi(t+x)}{\chi^+}\right]$$

$$\chi^{\pm} = \mathrm{ch}\,\pi(t \pm x) + \cos\pi y$$

which, apart from the notation, agrees with the result obtained in /12/.

If the function $f(x)$ approaches a constant value f_0 differing from zero as $x\to\infty$, we must add to the formula (1.5) obtained the solution of the uniformly compressed half-strip

$$\psi(x, y) = G(1-\nu)^{-1} f_0 x^2$$

We shall now examine the problem for a half-strip in the antisymmetric case (the bending case), when the boundary conditions take the form

$$x = 0,\ \tau_{xy} = 0,\ \sigma_x = 0 \tag{1.6}$$

$$y = \pm 1,\ \tau_{xy} = 0,\ u_y = f(x)$$

By reasoning similar to that used above, we obtain the following representation for the Airy function:

$$\psi(x, y) = -\frac{2G}{1-\nu}\int_0^{\infty} f'(t)\,\Psi(x, y, t)\,dt \tag{1.7}$$

$$\Psi(x, y, t) = \frac{1}{2\pi}\left[\frac{t-x}{2}\ln\frac{\chi_1^+\chi_2^-}{\chi_1^-\chi_2^+} + \frac{2\pi xt}{\chi^+}\sin\frac{\pi}{2}y\,\mathrm{sh}\frac{\pi}{2}(t+x)\right]$$

$$\chi_1^{\pm} = \mathrm{ch}\frac{\pi}{2}(t+x) \pm \sin\frac{\pi}{2}y, \quad \chi_2^{\pm} = \mathrm{ch}\frac{\pi}{2}(t-x) \pm \sin\frac{\pi}{2}y$$

In the case of a generalized plane stress state, in (1.5), (1.7), $2G/(1-\nu)$ must be replaced by E.

It is obvious that a solution to the problem of arbitrarily specified displacements of the front boundaries can be represented in the form of the summation of the symmetric and antisymmetric solutions.

2. Consider the first, fundamental boundary value problem for the half-strip, when a given load acts on its front. For simplicity we shall confine ourselves to the symmetric case with the following boundary conditions:

$$x = 0,\ \tau_{xy} = 0,\ \sigma_x = 0 \tag{2.1}$$

$$y = \pm 1,\ \tau_{xy} = 0,\ \sigma_y = g(x)$$

The problem reduces to deriving a function $f(x)$ from an integral equation of the first kind in the segment $(0, \infty)$, which is obtained from (1.5) when $y = 1$ by applying to it the operation $\partial^2/\partial x^2$. We shall separate the operator in this integral equation, corresponding to the problem for an infinite strip, and invert it. Finally we arrive at the relation

$$f(x)=G(x)-\int_0^\infty f'(\tau)\,d\tau\int_0^\infty K(x,t)\frac{\partial}{\partial t}\left[\frac{t\tau}{\operatorname{ch}\pi(t+\tau)-1}\right]dt \tag{2.2}$$

$$K(x,t)=2\int_0^\infty \frac{L(u)}{u}\cos ux\cos ut\,du$$

$$G(x)=\frac{1-\nu}{\pi G}\int_0^\infty g(t)\,K(x,t)\,dt,\quad L(u)=\frac{\operatorname{ch}2u-1}{\operatorname{sh}2u+2u}$$

Further, we consider the relation

$$[\operatorname{ch}\pi(t+\tau)-1]^{-1}=2\sum_{k=1}^\infty ke^{-\pi k(t+\tau)}$$

we integrate with respect to τ, using (1.3), after which we obtain

$$f(x)=G(x)-8(1-\nu)\sum_{k=1}^\infty(-1)^k kD_k\int_0^\infty L(u)\frac{u\cos ux}{(\pi^2k^2+u^2)^2}\,du \tag{2.3}$$

We differentiate (2.3) and multiply scalarly by $x\exp(-\pi nx)$. Consequently the problem of deriving the unknown function $f(x)$ can be reduced to the following system of linear algebraic equations in the coefficients D_n:

$$D_n=g_n+\sum_{k=1}^\infty c_{nk}D_k,\quad g_n=\frac{\pi n(-1)^n}{1-\nu}\int_0^\infty G'(x)\,xe^{-\pi nx}\,dx \tag{2.4}$$

$$c_{nk}=16\pi^2n^2k(-1)^{n+k}\int_0^\infty\frac{L(u)\,u^3\,du}{(\pi^2n^2+u^2)^2(\pi^2k^2+u^2)^2};\quad n=1,2,\ldots$$

We note that the 'degenerate' solution of (2.4) $D_n=g_n$ corresponds to the problem for an infinite strip.

A similar system for a problem related to the one examined, is obtained another way in /2/, where the apparatus of Koyalovich infinite systems was used to investigate it. From the results obtained in /2/ it follows that for a fairly smooth function $g(x)$ system (2.4) is regular, and we can apply the reduction method to it; the following relation holds

$$\lim_{n\to\infty}(-1)^nD_nn=a_0=\text{const} \tag{2.5}$$

It was also shown in /2/ that when the boundary values satisfy the condition of pairing of shear stresses in the area's corner point (as is the case in the problem under consideration), all the stresses and displacements at this point are finite.

By finding the coefficients D_n and substituting them into (2.3), we can determine the function $f(x)$, after which we determine the distribution of the stresses in the half-strip using (1.5). In particular, we have

$$\sigma_y=\sigma_y^\circ-2G\sum_{n=1}^\infty D_nb_n(x,y) \tag{2.6}$$

$$b_n(x,y)=(\pi nx-1)\,e^{-\pi nx}\cos\pi ny+I^+(x,y)$$

$$\sigma_x=\sigma_x^\circ+2G\sum_{n=1}^\infty D_nc_n(x,y) \tag{2.7}$$

$$c_n(x,y)=(\pi nx+1)\,e^{-\pi nx}\cos\pi ny-I^-(x,y)$$

$$I^\pm(x,y)=8n(-1)^n\int_0^\infty\frac{\operatorname{sh}u\operatorname{ch}uy\pm(\operatorname{ch}u\operatorname{ch}uy-y\operatorname{sh}u\operatorname{sh}uy)}{\operatorname{sh}2u+2u}\,\frac{u^2\cos ux}{(\pi^2n^2+u^2)^2}\,du$$

Stresses corresponding to the problem for an infinite strip are denoted by $^\circ$.

It is interesting to check the feasibility of the boundary conditions in the formulas for the stresses. If we put $y=\pm1$ in (2.6) and evaluate the integral occurring in the representation for b_n, it is easy to obtain $b_n(x,\pm1)\equiv0$. Hence, (2.6) automatically satisfies the last of the boundary conditions (2.1), irrespective of which values the coefficients D_n take. It can also be shown that the condition of half-strip equilibrium is automatically satisfied, i.e.

$$\int_0^\infty\sigma_y\,dx=\int_0^\infty g(x)\,dx,\quad -1\leqslant y\leqslant1$$

Unlike this formula (2.7) does not automatically make the stress σ_x vanish when $x=0$, and this boundary condition must be satisfied when the true values of the coefficients D_n are substituted into (2.7)

We shall investigate the rate of decay of the boundary layers. Using the theory of residues, one can show that $b_n(x,y)$ and $c_n(x,y)$ decrease as $x\to\infty$, as $x^a\exp(-\delta_nx)$ ($a>0$ is a certain constant). Here δ_n takes the values of the imaginary part of the zero function $\operatorname{sh}2u+2u$, lying in the upper half-plane, and also values equal to πn, $n=1,2,\ldots$ Since, as was noted in the introduction, the half-strip problem under consideration asymptotically covers the theory of plates, the established nature of the rate of decay of the boundary layers also remains true for plates with an arbitrary, smooth, lateral surface. The result obtained agrees withthe results obtained in /5,6/.

3. We shall employ the procedure developed above for the half-strip for an asymptotic

analysis of the two-dimensional problem for a narrow rectangle $(-1/\lambda \leqslant x \leqslant 1/\lambda,\ -1 \leqslant y \leqslant 1)$, where λ is a small parameter.

For a small thickness of the area, the boundary later in the vicinity of the rigt-hand lateral side $x = 1/\lambda$ obviously has only a slight effect on the left-hand boundary layer in the vicinity of the side $x = -1/\lambda$, and vice versa. Therefore, to construct boundary-layer solutions to a first approximation it is sufficient to examine the problem for two half-strips $(-1/\lambda \leqslant x < \infty)$ and $(-\infty < x \leqslant 1/\lambda)$. Using the idea employed for the first time in contact problems for a thin layer /14,15/, we shall represent the true solution to the problem asymptotically in the form

$$\varphi(x) = \varphi_1\left(\frac{1}{\lambda} + x\right) + \varphi_2\left(\frac{1}{\lambda} - x\right) - v(x), \quad |x| \leqslant \frac{1}{\lambda} \tag{3.1}$$

Here $\varphi(x)$ is any of the stresses or displacements in the rectangle, φ_1 is the analogous function in the first half-strip, φ_2 is in the second half-strip, and $v(x)$ is in the infinite strip $(-\infty < x < \infty)$. The error of representation (3.1) has the order of exp $(-\varepsilon/\lambda)$ as $\lambda \to 0\ (\varepsilon > 0)$, because as shown in /16,17/, the boundary layer parts of the solution to the problem in a thin rectangle decay exponentially as one moves inside the area.

As an example, we will examine the following problem. Let $\lambda = 1,\ g(x) = -A\cos\pi x$. This problem in the case of a rectnagular area was examined in /2/ by another method; here, however, it enables us to estimate the range of variation of the parameter λ, in which the proposed asymptotic approach is fairly effective.

Obviously, the series in (2.6), (2.7) (allowing for the relation (2.5)) has a high rate of convergence everywhere, apart from the area of the half-strip's lateral face $x = 0$. To compute σ_y when $x = 0$ in (2.6) instead of D_n the representation (2.4) was substituted and after summation with respect to n a series in k was obtained with a general term, which decreased as k increased, as $1/k^4$. To compute σ_x in the neighbourhood of the end $x = 0$ in (2.7) we took the finite number D_n, and further substituted the asymptotic formula (2.5), after which the series was summed in finite form.

To obtain numerical results a program in FORTRAN for the EC-1022 computer was compiled. The procedure for evaluating the integrals which occur in the coefficients b_n and c_n (2.6) and (2.7), and also in (2.4), are taken from /18/.

In Table 1 the value of the stresses $\sigma_y^* = A^{-1}10^3\sigma_y$ and $\sigma_x^* = A^{-1}10^3\sigma_x$ in the half-strip when $y = 0$ are compared with the corresponding values $\sigma_y^{\circ *}$ and $\sigma_x^{\circ *}$ for an infinite strip. Note that at a distance from the end of the order of the plate thickness they differ by less than 4%. This enables the rate of decay of the boundary-layer effect due to the zeroth boundary conditions on the end of the half-strip to be estimated. In the example considered, σ_y° and σ_x° can be expressed in terms of elementary functions.

Table 1

x	0.0	0.1	0 3	0.5	0.7	0.9	1.1	1.3	1.5	1.7	2.0
σ_y^*	640	551	288	−6,2	−254	−393	−388	−251	−33	182	338
$\sigma_y^{\circ *}$	350	333	206	0	−206	−333	−333	−206	0	206	350
σ_x^*	0	10	75	161	236	267	237	148	25	−92	−176
$\sigma_x^{\circ *}$	−182	−173	−107	0	107	173	173	107	0	−107	−182

Table 2

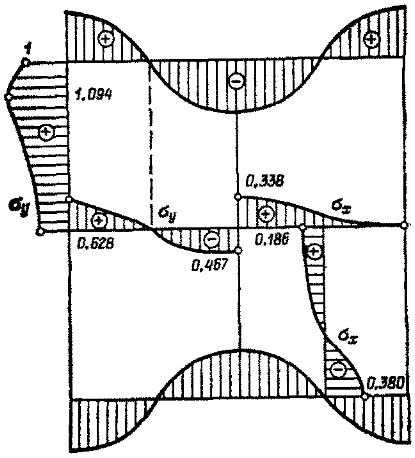
y	$x = 0$	0,5	1
0	−467 (−465.7)	−39.5 (−41,0)	628 (604,8)
	338 (334.0)	186 (180.5)	−3.1 (−0.7)
0.5	−649 (−647.9)	−48.7 (−50.2)	899 (893.7)
	171 (171.3)	61.3 (61,0)	−2,7 (0.6)
1	−1000 (−1000)	0 (0)	1000 (1001.7)
	−1172 (−1163.7)	−380 (−371,5)	— (0.2)

In Table 2 the values of the stresses σ_y^* obtained from (3.1) in the square area examined and represented by the odd rows are compared with the similar values obtained in /2/ (the latter are shown in brackets). A similar comparison for the stresses σ_x^* is represented by the even rows. The calculations were carried out retaining 5,10 and 18 terms in the infinite system (2.4) and also in all infinite series. The results shown in Table 2 are practically identical for all three cases.

Note that the values given in Table 2 include the error in the solution of the boundary value problem for a half-strip and the error of (3.1), which hold asymptotically only for thin regions. Nevertheless, the comparison indicates the effectiveness of the method for regions whose thickness is commensurate with their length.

Distribution curves of the stresses inside the square are shown in the figure.

The time taken to calculate any of the stresses σ_x and σ_y at an arbitrary point of the region (apart from the corner) averages 10-25 sec.



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